ON THE FORMULATION OF THE FIRST BOUNDARY VALUE PROBLEM FOR AN AXISYMMETRICALLY LOADED BODY OF REVOLUTION

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The problem of expressing the axisymmetric state of stress in a body of revolution in terms of p-analytic functions of a complex variable reduces to the boundary value problem which was formulated by Polozhii in the form

$$R^* + iZ^* = -i\left[2r\,\frac{\partial\overline{\Phi(\xi)}}{\partial r} - 2r\,\frac{\partial\Phi(\xi)}{\partial r} - 2z\,\frac{\partial\Phi(\xi)}{\partial z} + \overline{\Phi(\xi)} + \Psi(\xi)\right]_L + 2\mu\int_L^{\pi}\frac{u}{r}\,dz$$

$$R^* = \int_L R_n\,ds, \qquad Z^* = \int_L rZ_n\,ds \tag{0.1}$$

Here, $\Phi(\xi)$, and $\Psi(\xi)$ are arbitrary p-analytic functions of the complex variable $\xi = r + iz$ with characteristic $1/r$.

In [2], the boundary conditions in the above form were applied to problems concerned with symmetric states of stress in thick, infinite plates for which the integral of the displacement u is zero.

Below, we present a formulation of the first boundary value problem for the general case of an axisymmetric state of stress in a body of revolution. The formulation of Polozhii is based on Papkovich's general solution [3], which was expressed in terms of p-analytic functions of a complex variable.

1. The proposed formulation is based on the general biharmonic solutions of Love [4] and Grodskii [5]. Let us write the necessary formulas for stresses and displacements. Corresponding to Love's general solution satisfying the equation

$$\triangle\triangle\chi = 0, \qquad \triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \tag{1.1}$$

we have the following expressions:

For displacements

$$\frac{E}{1+\mu} u = \frac{\partial^2 \chi}{\partial r\, \partial z}, \qquad \frac{E}{1+\mu} w = -(1-2\mu)\, \triangle\chi - \frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial \chi}{\partial r} \tag{1.2}$$

For stresses

$$\sigma_r = -(1-\mu)\frac{\partial}{\partial z}(\triangle\chi) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial z^2}\right)$$

$$\sigma_\theta = -(1-\mu)\frac{\partial}{\partial z}(\triangle\chi) + \frac{\partial}{\partial z}\left(\frac{\partial^2\chi}{\partial r^2} + \frac{\partial^2\chi}{\partial z^2}\right) \tag{1.3}$$

$$\sigma_z = (1-\mu)\frac{\partial}{\partial z}(\triangle\chi) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial r^2}\right), \qquad \tau_{rz} = (1-\mu)\frac{\partial}{\partial r}(\triangle\chi) - \left(\frac{\partial}{\partial z}\frac{\partial^2\chi}{\partial r\,\partial z}\right)$$

Corresponding to Grodskii's general solution of the equation

$$DD\Omega = 0, \qquad D = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$$

we have the following expressions:

For displacements

$$\frac{E}{1+\mu} u = \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial^3\Omega}{\partial z^3} - \mu D\Omega\right), \qquad \frac{E}{1+\mu} w = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Omega}{\partial r}\right) - \mu D\Omega\right] \tag{1.4}$$

For stresses

$$\sigma_r = \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\frac{\partial^2\Omega}{\partial r\,\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z}\left(\frac{\partial^2\Omega}{\partial z^2} - \mu D\Omega\right), \qquad \sigma_\theta = \frac{1}{r^2}\frac{\partial}{\partial z}\left[\frac{\partial^2\Omega}{\partial z^2} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}D\Omega\right)\right]$$

$$\sigma_z = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2\Omega}{\partial r\,\partial z}\right)\right], \qquad \tau_{rz} = -\frac{\partial^2}{\partial r\,\partial z}\left(\frac{1}{r}\frac{\partial^2\Omega}{\partial r\,\partial z}\right) \tag{1.5}$$

2. Let us express the biharmonic functions X and Ω in terms of the harmonic vectors φ_1, and φ_2, which satisfy the equations

$$D(r\varphi_1) = 0, \qquad \triangle\varphi_2 = 0 \tag{2.1}$$

and for which the following relations [3] hold

$$\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_1) = \frac{\partial\varphi_2}{\partial z}, \qquad \frac{\partial\varphi_1}{\partial z} = -\frac{\partial\varphi_2}{\partial r} \tag{2.2}$$

For Love's solution we assume that

$$\triangle\chi = 2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_1) + \frac{\partial\varphi_2}{\partial z}\right] = 2\,\mathrm{div}\,(\varphi_1, \varphi_2) \tag{2.3}$$

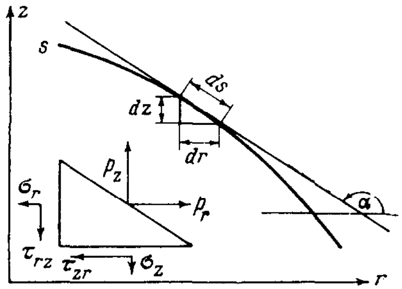
Then using (2.1) we obtain

$$2\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_1) + D(r\varphi_1) = \triangle(r\varphi_1), \quad 2\frac{\partial\varphi_2}{\partial z} + z\triangle\varphi_2 = \triangle(z\varphi_2) \tag{2.4}$$

Combining these equations and taking into account (2.3), we obtain

$$\Delta\chi = \Delta(r\varphi_1 + z\varphi_2) \tag{2.5}$$

From which the integration yields



$$\chi = r\varphi_1 + z\varphi_2 + \gamma_2 \tag{2.6}$$

Here γ_2 is an arbitrary function satisfying the equation $\Delta\gamma_2 = 0$.

To express Grodskii's solution in terms of φ_1, and φ_2, let

$$D\Omega = 2r\left(\frac{\partial\varphi_1}{\partial z} - \frac{\partial\varphi_2}{\partial r}\right) \equiv 2r\,\mathrm{curl}(\varphi_1, \varphi_2) \tag{2.7}$$

Then, with the aid of (2.1), we obtain, analogously to (2.4)

$$\begin{aligned} 2\frac{\partial}{\partial z}(r\varphi_1) + zD(r\varphi_1) &= D(rz\varphi_1) \\ 2r\frac{\partial\varphi_2}{\partial r} + r^2\triangle\varphi_2 &= D(r^2\varphi_2) \end{aligned} \tag{2.8}$$

Subtracting the second of these equations from the first and taking into account (2.7), we obtain

$$D\Omega = D\,[r(z\varphi_1 - r\varphi_2)] \tag{2.9}$$

which on integration yields

$$\Omega = r(z\varphi_1 - r\varphi_2 + \gamma_1) \tag{2.10}$$

Here γ_1 is an arbitrary function satisfying the equation $D(r\gamma_1) = 0$.

3. We now proceed to the formulation of the boundary value problem for the stress functions X and Ω. Consider a boundary element; the following equilibrium equations, interrelating the applied stresses p_r and p_z and the internal stresses, σ_r, σ_z, and τ_{rz} must hold (see figure):

$$p_r = \sigma_r \sin\alpha + \tau_{rz}\cos\alpha, \qquad p_z = \tau_{rz}\sin\alpha + \sigma_z\cos\alpha \tag{3.1}$$

In view of the fact that, on the boundary, $\sin\alpha = dz\,/\,ds$, and $\cos\alpha = -\,dr\,/\,ds$, we obtain

$$p_r = \sigma_r\frac{dz}{ds} - \tau_{rz}\frac{dr}{ds}, \qquad p_z = \tau_{rz}\frac{dz}{ds} - \sigma_z\frac{dr}{ds} \tag{3.2}$$

Substitution of (1.3) into the preceding equations leads to the boundary conditions for $X(s)$. Thus, making use of (2.2) and (2.3) in which it is assumed that

$$\triangle\chi = 4\frac{1}{r}\frac{\partial}{\partial r}(r\varphi_1) = 4\frac{\partial\varphi_2}{\partial z} \tag{3.3}$$

we obtain

$$\begin{aligned} p_r &= \frac{d}{ds}\left(\frac{\partial^2\chi}{\partial z^2}\right) + \frac{1}{r}\frac{\partial^2\chi}{\partial r\,\partial z}\frac{dz}{ds} - 4(1-\mu)\frac{d}{ds}\left(\frac{\partial\varphi_2}{\partial z}\right) \\ p_z &= \frac{1}{r} - \frac{d}{ds}\left(r\frac{\partial^2\chi}{\partial r\,\partial z}\right) + 4(1-\mu)\frac{1}{r}\frac{d}{ds}\left(r\frac{\partial\varphi_2}{\partial r}\right) \end{aligned} \tag{3.4}$$

Integration of the preceding relations along the contour yields

$$R \equiv \int_L p_r\, ds = \left[\frac{\partial^2\chi}{\partial z^2} - 4(1-\mu)\frac{\partial\varphi_2}{\partial z}\right]_L + \left[\frac{1}{r}\frac{\partial\chi}{\partial r}\right]_L$$
$$Z \equiv \int_L r p_z\, ds = -\left[\frac{\partial^2\chi}{\partial r\,\partial z} - 4(1-\mu)\frac{\partial\varphi_2}{\partial r}\right]_L \tag{3.5}$$

Substitution of these results into the expressions for the components of the principal external stress vector normal and tangential to the contour, leads to

$$N = R\frac{dz}{ds} - Z\frac{dr}{ds}, \quad T = Z\frac{dz}{ds} + R\frac{dr}{ds}$$
$$N = \left[\frac{d}{ds}\left(\frac{\partial\chi}{\partial z}\right) - 4(1-\mu)\frac{d\varphi_2}{ds} + \frac{1}{r}\frac{\partial\chi}{\partial r}\frac{dz}{ds}\right]_L \tag{3.6}$$

$$T = \left[\frac{d}{dn}\left(\frac{\partial\chi}{\partial z}\right) + 4(1-u)\frac{d\varphi_2}{dn} + \frac{1}{r}\frac{\partial\chi}{\partial r}\frac{dr}{ds}\right]_L, \qquad \frac{d}{dn} = \frac{\partial}{\partial r}\frac{dz}{ds} - \frac{\partial}{\partial z}\frac{dr}{ds} \tag{3.7}$$

The first of these equations is now no longer integrable along the contour.

Similarly, we may obtain expressions for the boundary conditions in connection with the stress function Ω. Using of (1.5), (2.2), (2.7) and (3.2), we find that

$$D\Omega = 4r\frac{\partial\varphi_1}{\partial z} = -4r\frac{\partial\varphi_2}{\partial r}, \quad p_r = \frac{\partial^2\Omega'}{\partial z^2}\frac{dz}{ds} + \frac{\partial^2\Omega'}{\partial r\,\partial z}\frac{dr}{ds} + \frac{1}{r}\frac{\partial\Omega'}{\partial r}\frac{dz}{ds} - \frac{4(1-\mu)}{r}\frac{\partial^2\varphi_1}{\partial z^2}\frac{dz}{ds}$$
$$p_z = -\frac{\partial^2\Omega'}{\partial r\,\partial z}\frac{dz}{ds} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Omega'}{\partial r}\right)\frac{dr}{ds}, \qquad \left(\Omega' = \frac{1}{r}\frac{\partial^2\Omega}{\partial r\,\partial z}\right) \tag{3.8}$$

Utilization of these expressions and integration of the components of the principal vector of the applied stresses, results in (3.9)

$$R \equiv \int_L p_r\, ds = \left[\frac{\partial\Omega'}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Omega}{\partial r}\right) - \frac{4(1-\mu)}{r}\frac{\partial\varphi_1}{\partial z}\right]_L, \quad Z = \frac{1}{r}\int_L r p_z\, ds = -\left[\frac{\partial\Omega'}{\partial r}\right]_L$$

Use of the expressions in (3.6) for N and T yields

$$N = \left[\frac{d\Omega'}{ds} - \frac{1}{r^2}\frac{\partial^2\Omega}{\partial z^2}\frac{dz}{ds} + \frac{4\mu}{r}\frac{\partial\varphi_1}{\partial z}\frac{dz}{ds}\right]_L, \quad T = \left[\frac{d\Omega'}{dn} - \frac{1}{r^2}\frac{\partial^2\Omega}{\partial z^2}\frac{dr}{ds} + \frac{4\mu}{r}\frac{\partial\varphi_1}{\partial z}\frac{dr}{ds}\right]_L \tag{3.10}$$

The first equation in (3.10) may be integrated along the contour. Thus

$$M = \int_L N\, ds = \left[\Omega' - \frac{1}{r^2}\frac{\partial\Omega}{\partial z} + 4\mu\frac{\varphi_1}{r}\right]_L \tag{3.11}$$

The above, together with (3.7), (3.9) and (3.10), yields the following integral expression for M

$$M = \int_L ds \int_L p_r\, dz - \int_L \frac{ds}{r}\int_L r p_z\, dr \tag{3.12}$$

This completes the formulation of the first boundary value problem for an axisymmetrically loaded body of revolution. Furthermore, it may be shown that the function Ω' satisfies the equation

$$\triangle\triangle\Omega' = 0 \tag{3.13}$$

Hence

$$\Omega' \equiv \chi$$

4. We will now express the stress functions χ and Ω in terms of p-analytic functions of a complex variable [1] with the characteristic $1/r$

$$f(\xi) = \varphi_1 + i\varphi_2 \tag{4.1}$$

This is the terminology used in connection with a complex variable for which the conditions for the existence of a derivative at a point lead to (2.1) and (2.2).

The stress functions χ and Ω may now be expressed in terms of p-analytic functions of a complex variable in a manner which is completely analogous to the Kolosov-Muskhelishvili method for expressing stress functions in the plane problem in terms of general analytic functions [6 and 7].

The expressions for χ in the form given in (2.6) may be obtained in terms of $f(\xi)$ by letting

$$2\chi = \bar{\xi} f(\xi) + \xi \bar{f}(\bar{\xi}) + i\,[\bar{g}(\bar{\xi}) - g(\xi)] \qquad (g(\xi) = \gamma_1 + i\gamma_2) \tag{4.2}$$

Here $g(\xi)$ is an arbitrary p-analytic function.

It is convenient to write Ω in (2.10) as $r\omega$, where

$$\omega = z\varphi_1 - r\varphi_2 + \gamma_1 \tag{4.3}$$

Expressing it in terms of p-analytic functions, we obtain

$$2\omega = i\,[\xi f(\xi) - \bar{\xi}\bar{f}(\bar{\xi})] + g(\xi) + \bar{g}(\bar{\xi}) \tag{4.4}$$

The complex representation of the principal vector of the applied load is given by:

$$R + iZ = -i\left[\frac{\partial\chi'}{\partial r} + i\,\frac{\partial\chi'}{\partial z}\right]_L - 4(1-\mu)\,i\left[\frac{\partial\varphi_1}{\partial z} - i\,\frac{\partial\varphi_2}{\partial z}\right]_L + \left[\frac{1}{r}\,\frac{\partial\chi}{\partial r}\right]_L \qquad \left(\chi' = \frac{\partial\chi}{\partial z}\right) \tag{4.5}$$

when based on Love's solution for χ, taking into account (3.5), and by

$$R + iZ = -i\left[\frac{\partial\chi}{\partial r} + i\,\frac{\partial\chi}{\partial z}\right]_L - \left[\frac{1}{r^2}\,\frac{\partial^2\Omega}{\partial z^2}\right] + 4\mu\left[\frac{1}{r}\,\frac{\partial\varphi_1}{\partial z}\right] \tag{4.6}$$

when based on Grodskii's solution for the stress function Ω, taking into account (3.9) and (3.13).

Substituting (4.1), (4.2) and (4.4) into (4.5) and (4.6), we obtain

$$\begin{aligned} R + iZ = \Big\{ &-i\,[f(\xi) + \xi\bar{f}'(\bar{\xi})] - \frac{2i}{\xi+\bar{\xi}}\left[f'(\xi) - \bar{f}'(\bar{\xi}) + \frac{\bar{\xi}}{2} f''(\xi) + \frac{\xi}{2}\bar{f}''(\bar{\xi})\right] + \\ &+ \frac{8\mu}{\xi+\bar{\xi}}\, i\,[f'(\xi) - \bar{f}'(\bar{\xi})] + \bar{g}'(\bar{\xi}) + \frac{2}{\xi+\bar{\xi}}\,[g''(\xi) + \bar{g}''(\bar{\xi})]\Big\}_L \end{aligned} \tag{4.7}$$

$$\begin{aligned} R + iZ\vartheta = \Big\{ &[f'(\xi) + \bar{f}'(\bar{\xi}) - \xi\bar{f}''(\bar{\xi}) - 4(1-\mu)\bar{f}'(\bar{\xi})] - i\bar{g}''(\bar{\xi}) + \\ &+ \frac{1}{\xi+\bar{\xi}}\,[f(\xi) + \bar{f}(\bar{\xi}) + \bar{\xi}f'(\xi) + \xi\bar{f}'(\bar{\xi})] - \frac{i}{\xi+\bar{\xi}}\,[g'(\xi) - \bar{g}'(\xi)]\Big\}_L \end{aligned} \tag{4.8}$$

Similarly, relations (3.10) and (3.11) may also be expressed in terms of functions of a complex variable.

Thus, in view of the analogy between the general solutions of the plane and axisymmetric problems in the theory of elasticity, we have succeeded in formulating the first boundary value problem for an axisymmetric state of stress in a body of revolution.

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